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# Diffusion–annihilation in the presence of a driving field

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**Abstract.** We study the effect of an external driving force on a simple stochastic reaction–diffusion system in one dimension. In our model each lattice site may be occupied by at most one particle. These particles hop with rates $(1 \pm \eta)/2$ to the right and left nearest neighbouring site respectively if this site is vacant, and annihilate with rate one if it is occupied. We show that density fluctuations (i.e. the $m$th moments $\langle N^m \rangle$ of the density distribution at time $t$) do not depend on the spatial anisotropy $\eta$ induced by the driving field, irrespective of the initial condition. Furthermore, we show that if one takes certain translationally invariant averages over initial states (e.g. random initial conditions) even local fluctuations do not depend on $\eta$. In the scaling regime $t \sim L^2$ the effect of the driving can be completely absorbed in a Galilei transformation. We compute the probability of finding a system of $L$ sites in its stationary state at time $t$ if it was fully occupied at time $t_0 = 0$.

## 1. Introduction

Stochastic reaction–diffusion processes in one dimension have received a considerable amount of attention (for a brief review see e.g. [1]). Despite their simplicity they show a very rich behaviour and some of the results so far obtained are of experimental relevance [2]. Another reason for their popularity is the wide range of applicability: systems of this kind map to interface dynamics [3], to polymers in random media [4] or, quite close to every day life, to traffic problems [5]. This list is by no means exhaustive, but provides ample motivation for the study of such models. In addition, they map to well known problems in many-body physics, particularly to integrable vertex models. A considerable amount of exact and rigorous results have been obtained using this mapping [6].

Here we consider a model defined on a ring of $L$ sites with periodic boundary conditions where each lattice site may be occupied by at most one particle. These particles (denoted $A$) hop with rates $(1 \pm \eta)/2$ to the right or left nearest neighbouring site respectively if this site is vacant (denoted $\emptyset$), and annihilate with rate $\lambda$ if it is occupied:

$$A\emptyset \to \emptyset A \qquad (1+\eta)/2 \tag{1}$$

$$\emptyset A \to A\emptyset \qquad (1-\eta)/2 \tag{2}$$

$$AA \to \emptyset\emptyset \qquad \lambda. \tag{3}$$

(In this paper we study only $\lambda = 1$.) This model, which is closely related to zero temperature Glauber dynamics [7, 10], has been studied by a number of authors [10–14]. In the limit $\lambda = 0$ the model reduces to the well known asymmetric exclusion process [15], whereas for $\lambda \to \infty$, equivalent to the absence of any diffusion, the model becomes equivalent to random sequential adsorption of dimers [16]. A discrete time version of the model was studied in [17]. Physically, the exclusion principle corresponds to a hard-core on-site repulsion and

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the asymmetry in the left and right hopping rates may be thought of as the result of a field driving the particles in one preferred direction. The pair annihilation finally takes into account the possibility of an effective short-range attraction leading to unbreakable, inert pairs of particles.

For $\lambda = 0$ the effect of the driving field has been well studied and it turns out to be rather drastic. As opposed to the driven model without exclusion where the drift may be absorbed in a simple lattice Galilei transformation [18], here the asymmetry leads to the formation of shocks [19]. Not many exact results are known for the time evolution of the system, but the mapping to the six-vertex model has been shown to be useful by using the Bethe ansatz [20, 21] and related methods [22]. The dynamical exponent appearing in the dynamical structure function of the system was found to be $z = 3/2$ [20] rather than $z = 2$ in the undriven exclusion process or the non-interacting driven system without exclusion. For non-zero but small annihilation rates $\lambda$, a recent study based on scaling arguments, mean-field approaches, random walk considerations and numerical results has given a very different scenario: in the presence of pair annihilation the effect of the driving appears to be very small [14]. In addition, it is known that for $\lambda = 1$ and an initially full lattice neither the time-dependent density nor the two-point density correlation function depend on the driving [10].

The effect of a driving field has also been studied in other related models. Numerical evidence shows that in a similar reaction–diffusion process where exclusion particles of two different species $A$ and $B$ diffuse and annihilate ($A + B \to \emptyset$), a driving field does change the universality class if the two kinds of particles move in opposite directions [23]. These observations make a more detailed study of such systems desirable.

It is the aim of this paper to provide some exact results concerning the driving in the presence of the reaction for the one-species model and to go beyond the results achieved so far in [10, 14] which study the behaviour of the model only in terms of the (total) average density for random initial conditions and the two-point density correlation function for an initially full lattice. In section 2 we define the model in terms of a master equation written in a quantum Hamiltonian formalism. We shall focus on the case $\lambda = 1$ because, for that particular choice, the model may be described in terms of free fermions and treated rigorously. Exact expressions for various correlation functions (in principle, for *all* correlation functions) become readily available. It turns out (section 3) that fluctuations in the total density of particles do not depend on the driving field at all; the same is true for some other average values to be specified below. Furthermore, if one takes certain translationally invariant averages over initial states, e.g. random initial conditions, arbitrary *local* density correlations show no dependence on the asymmetry $\eta$. These results generalize the results of [10] which studies the average density and the two-point correlation function for an initially full lattice and other earlier findings [24]. The general solution of the master equation and some further results concerning the scaling regime are given in section 4. In section 5 we summarize and discuss our results.

## 2. The master equation in quantum Hamiltonian formulation

### *2.1. Definitions*

We define the process in terms of a master equation for the probability $f(\underline{n}; t)$ of finding, at time $t$, any configuration $\underline{n}$ of particles in the system of $L$ sites with periodic boundary conditions. Here $\underline{n} = \{n_1, n_2, \ldots, n_L\}$ where $n_i = 0, 1$ and $1 \leqslant i \leqslant L$ labels the sites of the lattice. An alternative possibility is to use the set $\{x_1, x_2, \ldots, x_N\}$ of occupied lattice

sites; in this notation, the empty set represents the empty lattice and $1 \leqslant N \leqslant L$ is the total number of particles in the configuration. We shall express the time evolution given by the master equation in terms of a quantum Hamiltonian $H$ [25]. This is discussed in detail in a number of publications (for consistency of notation see e.g. [6]) and we shall repeat only the essential elements of the mapping. The advantage of this approach is that there are standard methods of dealing with the resulting time evolution operator $H$. The applicability of these techniques, in the case at hand essentially a Jordan–Wigner transformation and the representation of states in terms of a fermionic Fock space, does not arise naturally and obviously if the master equation is written down in standard form.

The idea is to represent each of the $2^L$ possible configurations in $X = \{0, 1\}^L$ by a vector $|\underline{n}\rangle$ (or $|x_1, \ldots, x_N\rangle$, with $|0\rangle \equiv |\ \rangle$ being the empty state). The probability distribution is then mapped to a state vector

$$|f(t)\rangle = \sum_{\underline{n}\in X} f(\underline{n}; t)|\underline{n}\rangle. \tag{4}$$

The vectors $|\underline{n}\rangle$ together with the transposed vectors $\langle \underline{n}|$ form an orthonormal basis of $(\mathbb{C}^2)^{\otimes L}$ and the time evolution is defined in terms of a linear 'Hamilton' operator $H$ acting on this space of dimension $2^L$:

$$\frac{\partial}{\partial t}|f(t)\rangle = -H|f(t)\rangle. \tag{5}$$

A state at time $t = t_0 + \tau$ is, therefore, given in terms of an initial state at time $t_0$ by

$$|f(t_0 + \tau) = \mathrm{e}^{-H\tau}|f(t_0)\rangle. \tag{6}$$

From (4) and (5) and using $f(\underline{n}; t) = \langle \underline{n}\,|\,f(t)\,\rangle$ the master equation takes the form

$$\frac{\partial}{\partial t} f(\underline{n}; t) = -\langle \underline{n}|H|f(t)\rangle. \tag{7}$$

Note that

$$\langle s|f(t)\rangle = \sum_{\underline{n}\in X} f(\underline{n}; t) = 1 \tag{8}$$

where

$$\langle s| = \sum_{\underline{n}\in X}\langle \underline{n}| \tag{9}$$

which expresses conservation of probability and which implies $\langle s|H = 0$.

Expectation values $\langle Q\rangle$ are calculated as matrix elements of suitably chosen operators $Q$. A complete set of observables are the occupation numbers $n_k = 0, 1$. Defining projection *operators* on states with a particle on site $k$ of the chain as

$$n_k = \tfrac{1}{2}\left(1 - \sigma_k^z\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_k \tag{10}$$

one finds that the average density of particles at site $k$ is given by $\langle n_k\rangle = \langle s|n_k|f(t)\rangle$. Correlation functions $\langle n_{k_1}\cdots n_{k_j}\rangle$, i.e. the probabilities of finding particles on the set of sites $\{k_1, \ldots, k_j\}$, are computed analogously.

For later convenience we also introduce the operators $s_k^{\pm} = (\sigma_k^x \pm \mathrm{i}\sigma_k^y)/2$. In our convention

$$s_k^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_k \tag{11}$$

creates a particle at site $k$ when acting to the right, while

$$s_k^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_k \tag{12}$$

annihilates a particle at site $k$. In spin language this means that spin-down is identified with a particle, while spin-up represents a vacancy. Note that

$$\langle s|s_k^+ = \langle s|n_k \text{ and } \langle s|s_k^- = \langle s|(1-n_k). \tag{13}$$

Introducing the ladder operators $S^\pm = \sum_{k=1}^L s_k^\pm$ one may write

$$\langle s| = \langle 0|e^{S^+}. \tag{14}$$

Using the commutation relations for the Pauli matrices then yields (13).

Now we can define the process in terms of the quantum Hamiltonian

$$H = \sum_{k=1}^{L} u_k \tag{15}$$

with the nearest-neighbour reaction matrices

$$\begin{aligned} u_k = \frac{1+\eta}{2}\left(n_k(1-n_{k+1}) - s_k^+ s_{k+1}^-\right) + \frac{1-\eta}{2}\left((1-n_k)n_{k+1} - s_k^- s_{k+1}^+\right) \\ +\lambda(n_k n_{k+1} - s_k^+ s_{k+1}^+) \\ \overset{\lambda\to 1}{=} \frac{1+\eta}{2}(n_k - s_k^+ s_{k+1}^-) + \frac{1-\eta}{2}(n_{k+1} - s_k^- s_{k+1}^+). \end{aligned} \tag{16}$$

This together with (7) defines the process. The Hamiltonian may be written $H = H_s + \eta H_d$ where the driving part $H_d$ is given by $H_d = 1/2\sum_{k=1}^L (s_k^- s_{k+1}^+ - s_k^+ s_{k+1}^-)$ and $H_s$ is the Hamiltonian for the system without driving.

### 2.2. *Fermion representation of H*

For $\lambda = 1$ the Hamiltonian becomes bilinear in the creation and annihilation operators $s_k^\pm$. This suggests rewriting $H$ by introducing fermionic operators through a Jordan–Wigner transformation [26]. We define

$$Q_k = \prod_{i=1}^{k} \sigma_i^z \tag{17}$$

$$a_k^\dagger = s_k^- Q_{k-1} \tag{18}$$

$$a_k = Q_{k-1} s_k^+ \tag{19}$$

satisfying the anticommutation relations $\{a_k, a_l\} = \{a_k^\dagger, a_l^\dagger\} = 0$ and $\{a_k^\dagger, a_l\} = \delta_{k,l}$. Note that because of the periodic boundary conditions for the Pauli matrices one has $a_{L+1}^\dagger = a_1^\dagger Q_L$ and $a_{L+1} = Q_L a_1$. $Q_L$ may be written $Q_L = (-1)^N$ where $N = \sum n_k$ is the number operator. Since by the action of $H$ the particle number changes only in units of two, $Q_L$ commutes with $H$ and splits it into a sector with an even number of particles ($Q_L = +1$) and into a sector with an odd number of particles ($Q_L = -1$). In terms of the fermionic operators one finds

$$s_k^+ s_{k+1}^- = a_{k+1}^\dagger a_k \qquad s_k^- s_{k+1}^+ = a_k^\dagger a_{k+1} \tag{20}$$

$$s_k^+ s_{k+1}^+ = a_{k+1} a_k \qquad s_k^- s_{k+1}^- = a_k^\dagger a_{k+1}^\dagger \tag{21}$$

$$n_k = a_k^\dagger a_k \tag{22}$$

and we arrive at the following expressions for $H_s$ and $H_d$:

$$H_s = -\frac{1}{2}\sum_{k=1}^{L}\left\{a_{k+1}^{\dagger}a_k + a_k^{\dagger}a_{k+1} + 2a_{k+1}a_k - 2n_k\right\} - (\lambda - 1)\sum_{k=1}^{L}\{a_{k+1}a_k - n_k n_{k+1}\} \tag{23}$$

$$H_d = -\frac{1}{2}\sum_{k=1}^{L}\left\{a_{k+1}^{\dagger}a_k - a_k^{\dagger}a_{k+1}\right\}. \tag{24}$$

We also note that $H_s$ and $H_d$ commute if $\lambda = 1$. Independently of $\lambda$ and $\eta$ the non-degenerate ground state of $H$, corresponding to the steady state of the system, is the totally empty state $|0\rangle$ in the sector with an even number of particles, and the state $|0\rangle^{\text{odd}} = 1/L\sum_{k=1}^{L}|k\rangle$ where one particle may be found with equal probability $1/L$ anywhere in the lattice in the sector with an odd number of particles.

Since expectation values $\langle n_k(t)\rangle$ of the stochastic variables $n_k = 0, 1$ are given by the matrix elements $\langle s|n_k \exp(-Ht)|f\rangle$ of the operator $n_k$ and since also $\langle s|\exp(Ht) = \langle s|$, one may introduce time-dependent operators

$$\mathcal{O}(t) = \exp(Ht)\mathcal{O}\exp -(Ht) \tag{25}$$

and write $\langle\mathcal{O}(t)\rangle = \langle s|\mathcal{O}(t)|f\rangle$ for an arbitrary operator $\mathcal{O}$. According to definition (25) one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O} \equiv \dot{\mathcal{O}} = [H, \mathcal{O}]. \tag{26}$$

Among the quantities of interest are the density of particles in the system and fluctuations in this quantity. The $m$th moment of the particle number distribution is the expectation value $\langle N^m(t)\rangle$ where $N$ is the number operator $N = \sum_{k=1}^{L} a_k^{\dagger}a_k$. Any $\mathcal{O}$ may be written as a product of the fermionic annihilation and creation operators and it is, therefore, sufficient to study the time evolution of these operators.

It is useful to introduce the Fourier transforms

$$b_p = \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{L}}\sum_{k=1}^{L}\mathrm{e}^{2\pi\mathrm{i}kp/L}a_k \tag{27}$$

$$b_p^{\dagger} = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{L}}\sum_{k=1}^{L}\mathrm{e}^{-2\pi\mathrm{i}kp/L}a_k^{\dagger} \tag{28}$$

satisfying $\{b_p, b_q\} = \{b_p^{\dagger}, b_q^{\dagger}\} = 0$ and $\{b_p^{\dagger}, b_q\} = \delta_{p,q}$. Inverting (27) and (28) yields

$$a_k = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{L}}\sum_{p}\mathrm{e}^{-2\pi\mathrm{i}kp/L}b_p \tag{29}$$

$$a_k^{\dagger} = \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{L}}\sum_{p}\mathrm{e}^{2\pi\mathrm{i}kp/L}b_p^{\dagger}. \tag{30}$$

Thus, the representation of the number operator in Fourier space is

$$N = \sum_{p} b_p^{\dagger}b_p. \tag{31}$$

Here the sum runs over all integers $p = 0, \ldots, L-1$ in the sector with an odd number of particles and over the half-odd integers $p = 1/2, 3/2, \ldots, L - 1/2$ in the even sector.

The Hamiltonian for $\lambda = 1$ reads

$$H_s = \sum_p \left\{ \left(1 - \cos\frac{2\pi p}{L}\right) b_p^\dagger b_p + \sin\frac{2\pi p}{L} b_{-p} b_p \right\} \tag{32}$$

$$H_d = -\mathrm{i} \sum_p \sin\left(\frac{2\pi p}{L}\right) b_p^\dagger b_p. \tag{33}$$

$H_s$ has already been obtained in [13].

### 2.3. *Initial states in the fermion representation*

Having discussed the representation of operators in terms of the fermionic operators we turn to the representation of initial configurations or states. In what follows we shall use the term '(initial) configuration' for a vector $|k_1, \ldots, k_N\rangle$ or $|\underline{n}\rangle$, i.e. for a simultaneous eigenstate of all $n_k$. (Initial) states $|f\rangle = \sum_{\underline{n}} f(\underline{n})|\underline{n}\rangle$ are vectors which may be superpositions of such configurations, and normalized such that $\langle s|f\rangle = 1$. The vector $|0\rangle$ is the vacuum state with respect to the annihilation operators $a_k|0\rangle = 0$. In spin language this is the ferromagnetic state with all spins up. Acting with creation operators yields

$$a_{k_1}^\dagger \cdots a_{k_N}^\dagger |0\rangle = |k_1, \ldots, k_N\rangle \qquad (k_1 < k_2 < \cdots < k_N). \tag{34}$$

A general translationally invariant $N$-particle state is obtained by acting with products $b_{p_1}^\dagger \cdots b_{p_N}^\dagger$ on $|0\rangle$ where $\sum_i p_i = 0$. A special class are those built by polynomials in the bilinear expressions $B_p^\dagger = b_{-p}^\dagger b_p^\dagger$ where $p = 1/2, 3/2, \ldots, (L-1)/2$. Among these particular translationally invariant states are uncorrelated random initial conditions with an even number of particles. The uncorrelated random initial state with density $\rho$ is the product measure

$$|\rho\rangle = \begin{pmatrix} 1-\rho \\ \rho \end{pmatrix}^{\otimes L} = \sum_{N=0}^{L} \rho^N (1-\rho)^{L-N} |N\rangle \tag{35}$$

where $|N\rangle$ is the $N$-particle state where each configuration appears with equal weight. By projection on the sectors with even and odd particle numbers we obtain

$$|\rho\rangle^{\text{even(odd)}} = \frac{1 \pm Q_L}{1 \pm (1-2\rho)^L} |\rho\rangle \tag{36}$$

where we have used $Q_L$ defined in (17).

From the representations (32) and (33) of $H$ and using $(B_p)^2 = (B_p^\dagger)^2 = 0$ one finds for the left zero-energy eigenvector $\langle s|$ of $H$ that

$$\begin{aligned} \langle s| &= \langle 0| \prod_p \left(1 + \cot\left(\frac{\pi p}{L}\right) b_p b_{-p}\right) + \langle 0| b_0 \prod_{p'} \left(1 + \cot\left(\frac{\pi p'}{L}\right) b_{p'} b_{-p'}\right) \\ &= \langle 0| \exp\left[\sum_p \cot\left(\frac{\pi p}{L}\right) b_p b_{-p}\right] + \langle 0| b_0 \exp\left[\sum_{p'} \cot\left(\frac{\pi p'}{L}\right) b_{p'} b_{-p'}\right] \\ &\equiv \langle s|^{\text{even}} + \langle s|^{\text{odd}}. \end{aligned} \tag{37}$$

The product and sum over $p$, $p'$ run over $p = 1/2, 3/2, \ldots, (L-1)/2$ (even sector), and over $p' = 1, 2, \ldots, L/2 - 1$ (odd sector), respectively. This in turn implies

$$\frac{1}{N!}\left(\sum_p \cot\left(\frac{\pi p}{L}\right) B_p^\dagger\right)^N = \sum_{1 \leqslant k_1 < \cdots < k_{2N} \leqslant L} a_{k_1}^\dagger \cdots a_{k_{2N}}^\dagger. \tag{38}$$

Hence,

$$|2N\rangle = \frac{1}{N!}\left(\sum_p \cot\left(\frac{\pi p}{L}\right) B_p^\dagger\right)^N |0\rangle \tag{39}$$

$$|\rho\rangle^{\text{even}} = \frac{2}{1+(1-2\rho)^L} \prod_p \left((1-\rho)^2 + \rho^2 \cot\left(\frac{\pi p}{L}\right) B_p^\dagger\right)|0\rangle. \tag{40}$$

The completely full lattice is simply given by

$$|L\rangle = \prod_p B_p^\dagger |0\rangle. \tag{41}$$

In what follows we shall focus on the even sector. For a study of correlation functions (section 4) it is useful to note

$$\langle s|^{\text{even}} \left(b_p^\dagger + \cot\left(\frac{\pi p}{L}\right) b_{-p}\right) = 0 \tag{42}$$

which may be verified using the momentum space representation (37) of $\langle s|^{\text{even}}$.

## 3. Dynamics of the system on a translationally invariant subspace

In the last section we introduced the subspace $\mathcal{V}$ generated by the operators $B_p^\dagger = b_{-p}^\dagger b_p^\dagger$. Together with $B_p = b_p b_{-p}$ they satisfy the algebra of the Pauli matrices, with $[B_p, B_p^\dagger] = b_{-p}b_{-p}^\dagger - b_p^\dagger b_p \equiv 2C_p$ playing the role of the $\sigma_p^z$ matrix. $I_p \equiv b_p^\dagger b_p + b_{-p} b_{-p}^\dagger$ commutes with all $B_p^\dagger$, $B_p$, $C_p$ and acts as a unit operator on this subspace; it also satisfies $I_p X_p = X_p I_p = X_p$ for $X_p = B_p, B_p^\dagger, C_p$. These relations are easy to verify by using the anticommutation relations for $b_p^\dagger$ and $b_p$.

This subspace is of interest for three reasons. First, $H$ can be written in terms of these operators:

$$H = \sum_p \left(1 - \cos\left(\frac{2\pi p}{L}\right)\right) N_p - 2\sin\left(\frac{2\pi p}{L}\right) B_p - i\eta \sin\left(\frac{2\pi p}{L}\right)(I_p - 1) \tag{43}$$

where we have introduced the number operator $N_p = 1 - 2C_p$. Therefore, $\mathcal{V}$ is an invariant subspace of $H$. Second, a physically important class of initial conditions, namely random initial conditions, including the steady state and the fully occupied lattice, are in this subspace. Finally, some physically important expectation values are given by operators constructed from $B_p^\dagger$ and $B_p$. In particular, using (31) one obtains

$$N = \sum_p N_p. \tag{44}$$

Without further calculation we can now state the following results.

(i) The subspace $\mathcal{V}$ of dimension $2^{L/2}$ generated by $B_p^\dagger$ acting on the vacuum state $|0\rangle$ is an invariant subspace of $H$. On this subspace $H_d = 0$, i.e. the driving has no effect on any correlation function if the system is at time $t = 0$ in an initial state which is contained in $\mathcal{V}$ (e.g. random initial conditions).

This result can be seen by observing the fact that $I_p$ is the unit operator on this subspace which gives $H_d = 0$. As a result, the state at time $t$ does not depend on $\eta$ which in turns implies that no correlation in that state can depend on the driving.

(ii) The time evolution of the operators $\mathcal{O}$ built from operators $B_p^\dagger$ and $B_p$ (e.g. the density operator) does not depend on the driving, irrespective of the initial condition.

This is again obvious since $H_d$ commutes with any such operator. Applying this result to powers $N^m$ of $N$ we find the next result.

(iii) The probability $P(N;t)$ of finding precisely $N$ particles at time $t$ in the system does not, for any initial condition, depend on the driving.

The same also applies for the moments $\langle N^m(t)\rangle$ of this distribution. For $m=1$ this was shown in [10]. An interesting special case is $P_f(0;t)=\langle 0|f(t)\rangle$ giving the probability that the system has reached the steady state at time $t$ from an initial state $|f\rangle$. To obtain this quantity we note that $H$ (43) restricted to the subspace $\mathcal{V}$ is a sum of $2\times 2$ matrices $h_p$:

$$H=2\sum_p \begin{pmatrix} 0 & -\sin 2\pi p/L \\ 0 & 1-\cos 2\pi p/L \end{pmatrix}_p. \tag{45}$$

For random initial conditions (40) with an even number of particles, computing $\exp(-Ht)=\prod_p \exp(-h_p t)$ and taking the product of the matrix elements $\langle 0|B_p^\dagger(t)|0\rangle$ then gives the exact expression

$$P_\rho(0;t)=\frac{2}{1+(1-2\rho)^L}\prod_p\left[(1-\rho)^2+\rho^2\cot\left(\frac{\pi p}{L}\right)\left(1-\mathrm{e}^{-2t(1-\cos(2\pi p/L))}\right)\right]. \tag{46}$$

For an initially full lattice this simplifies to

$$P_1(0;t)=\prod_p\left(1-\mathrm{e}^{-2t(1-\cos(2\pi p/L))}\right)\overset{L\to\infty}{\sim}\mathrm{e}^{-L/\sqrt{4\pi t}}. \tag{47}$$

## 4. Full solution of the master equation

The master equation is solved if one has found explicit expressions for $a_k^\dagger(t)$ and $a_k(t)$, or their Fourier transforms. A knowledge of these quantities then allows the explicit calculation of any correlation function for any initial condition, including multi-time correlators giving arbitrary conditional probabilities. We will compute $a_k^\dagger(t)$ and $a_k(t)$ and study the effect of the driving on these quantities for late times. Applications to specific correlation functions of interest will be given elsewhere. For definiteness we only study the sector with an even number of particles and we also assume, as in the preceding section, $L$ even.

It is interesting to first study the differential equation satisfied by the local density $\langle n_k(t)\rangle$. Differentiating with respect to time one finds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle n_k(t)\rangle=&\frac{1}{2}\left(\langle n_{k+1}(t)\rangle+\langle n_{k-1}(t)\rangle-2\langle n_k(t)\rangle\right)-\frac{\eta}{2}\left(\langle n_{k+1}(t)\rangle-\langle n_{k-1}(t)\rangle\right)\\&-(\lambda+\eta)\langle n_{k-1}(t)n_k(t)\rangle-(\lambda-\eta)\langle n_k(t)n_{k+1}(t)\rangle.\end{aligned} \tag{48}$$

In the linear terms one recognizes a lattice Laplacian and lattice derivative respectively. For $\lambda=0$ the nonlinear term is also a lattice derivative and the equation is a discrete form of Burgers equation [27] describing the evolution of shocks for $\eta\neq 0$. Any $\lambda>0$ will result in a strong dampening of the amplitude and the question arises to which extent the nonlinear effects associated with the driving continue to play a role. For $\lambda=1$ this question was partially answered in the last section, the result being, somewhat surprisingly, that, for the initial conditions considered, the driving induces no nonlinear effects at all. Here we study the form of local correlations for arbitrary initial conditions.

The simple form of $H$ in Fourier space suggests studying $b_p^\dagger(t)$ and $b_p(t)$ rather than $a_k^\dagger(t)$ and $a_k(t)$. Applying (26) gives a set of two coupled ordinary differential equations†:

$$\frac{\mathrm{d}}{\mathrm{d}t}b_p^\dagger(t) = \epsilon_p b_p^\dagger(t) + 2\sin\left(\frac{2\pi p}{L}\right)b_{-p}(t) \tag{49}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}b_p(t) = -\epsilon_p b_p(t) \tag{50}$$

solved by

$$b_p^\dagger(t) = \mathrm{e}^{\epsilon_p t}\left(b_p^\dagger + \cot\frac{\pi p}{L}\left(1-\mathrm{e}^{-(\epsilon_p+\epsilon_{-p})t}\right)b_{-p}\right) \tag{51}$$

$$b_p(t) = \mathrm{e}^{-\epsilon_p t}b_p \tag{52}$$

with $b_p^\dagger(0) = b_p^\dagger$, $b_p(0) = b_p$ and

$$\epsilon_p = 1 - \cos\frac{2\pi p}{L} - \mathrm{i}\eta\sin\frac{2\pi p}{L}. \tag{53}$$

From (51) and (52) one obtains

$$n_k(t) = \frac{1}{L}\sum_{p,p'}\mathrm{e}^{2\pi\mathrm{i}k(p-p')/L}b_p^\dagger(t)b_{p'}(t) \tag{54}$$

and, therefore, an explicit expression of any correlation function at time $t$ in terms of correlators in the initial state. In particular, one may use (42) to obtain

$$\langle n_k(t)\rangle = \frac{1}{L}\sum_{p,p'}\mathrm{e}^{2\pi\mathrm{i}k(p-p')/L-(\epsilon_{-p}+\epsilon_{p'})t}\cot\frac{\pi p}{L}\langle b_{p'}b_p\rangle. \tag{55}$$

This solves the initial value problem for the average density.

Returning to arbitrary time-dependent correlators we note that (51)–(53) demonstrate the impact of the driving on the system in the scaling regime. For times $t \gg L^2$ all correlations decay exponentially with correlation time $\tau = 2L^2/\pi^2$ resulting from the slowest mode $p = 1/2$. For times in the scaling regime $t \sim L^2$ and $L$ large, one may approximate $\epsilon_p$ by

$$\epsilon_p \approx -\eta\frac{2\pi\mathrm{i}}{L}p + \frac{2\pi^2}{L^2}p^2 \tag{56}$$

that is, the effect of the driving may be absorbed in a Galilei transformation $r_i \to r_i + \eta t$ where $r_i = k_i/L$ are the scaled space coordinates appearing in the correlation function. There are no other effects, unlike in the absence of annihilation where the driving induces the evolution of shocks. For *arbitrary* translationally invariant initial conditions the dependence of the correlation functions on $\eta$ vanishes completely in the scaling limit. It may be worthwhile pointing out that in the continuum theory this is not surprising: the two-point correlator proportional to $\lambda$ appearing in (48) (which in the continuum limit becomes an additional term proportional to $n^2$ in Burgers equation) indeed suggests that for long times the nonlinearity induced by the driving becomes irrelevant.

† Because of the boundary conditions these equations describe the time evolution of the creation and annihilation operators only when applied to products with an even number of operators. This is not a restriction as all expectation values $\langle n_{k_1}\dots n_{k_N}\rangle$ are of this form.

## 5. Conclusions

We have studied the effect of driving in a simple reaction–diffusion system where exclusion particles hop with rates $(1 \pm \eta)/2$ to the right and left respectively if the nearest-neighbour sites are empty, and which are annihilated in pairs with rate one if the nearest-neighbour site is occupied. We obtained the following results.

(i) Certain translationally invariant time-dependent expectation values including the $m$th moments $\langle N^m \rangle$ of the particle number distribution do not depend on the driving parameter $\eta$, regardless of the initial condition.

(ii) Arbitrary time-dependent expectation values of occupation numbers $\langle n_{k_1} \dots n_{k_N} \rangle$ do not depend on $\eta$ if at time $t = 0$ one takes certain translationally invariant averages over initial states, e.g. uncorrelated random initial conditions.

(iii) In the scaling regime $t \approx L^2$, the effect of the driving can be completely absorbed in a Galilei transformation.

(iv) The probability that the system has reached its steady state (the empty lattice) at time $t$ from an uncorrelated random initial condition with density $\rho$ and an even number of particles is given by (46).

These results have been obtained for an annihilation rate $\lambda = 1$. It would be very interesting to study the system for other values of $\lambda$ in order to understand the transition to the limiting cases $\lambda = 0$ and $\lambda = \infty$. For $\lambda = 0$ (the asymmetric exclusion process) the introduction of driving has a very strong effect, it causes the evolution of shocks from local inhomogeneities and it is not clear to which extent these effects survive in the presence of annihilation; our calculation shows that for $\lambda = 1$ they are completely absent in the scaling regime. It is interesting to note that it is not the exclusion principle, as such, which is responsible for the nonlinear behaviour of the asymmetric exclusion process, but the strength of the pair interaction between neighbouring particles. In stochastic language this is the rate of change of a pair of neighbouring particles $\lambda$ compared to the diffusion rates $(1 \pm \eta)/2$. A satisfactory understanding of this interplay remains an open problem. Intuitively one would expect the nonlinear behaviour to vanish for late times for any $\lambda \neq 0$ as the system will then be almost empty and, therefore, effectively non-interacting. This is supported by various arguments put forward in [14] and by the known dynamical exponent $z = 2$ of the asymmetric exclusion process in the regime of low (infinitesimal) densities [12] (as opposed to $z = 3/2$ for finite densities). A qualification of the expression 'late times', and, more importantly, what happens before such late times remains open to debate.

Two other interesting open problems are (a) the generalization of our results to the more general model which also allows for the creation of pairs of particles [10] and (b) the approach to local equilibrium, i.e. to a local extended region of empty sites. The first question to be asked in this context is what is the probability of reaching a state with, say, $M$ empty neighbouring sites. The next question is then how this state further evolves in time. It is, of course, not stationary as particles are injected and absorbed by diffusion at the boundaries of this region, but with time-dependent (vanishing) rates. For this problem the driving will make a difference as the injection of particles at the left boundary of this region will be stronger than the loss of particles (if particles move preferredly to the left). At the right boundary of this region the situation will be reversed, the loss will exceed the gain. The result of section 4 shows that the problem with a drift may be obtained from the symmetric problem by a Galilei transformation and, therefore, gives a partial answer to the question. Equation (46) gives the probability of finding the system in global equilibrium which does not depend on the driving.

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